

# Condition for the Superradiance Modes in Higher-Dimensional Rotating Black Holes with Multiple Angular Momentum Parameters

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(February 1, 2008)

## Abstract

The condition for the existence of the superradiance modes is derived for the incident scalar, electromagnetic and gravitational waves when the spacetime background is a higher-dimensional rotating black hole with multiple angular momentum parameters. The final expression of the condition is  $0 < \omega < \sum_i m_i \Omega_i$ , where  $\Omega_i$  is an angular frequency of the black hole and,  $\omega$  and  $m_i$  are the energy of the incident wave and the  $i$ -th azimuthal quantum number. The physical implication of this condition in the context of the brane-world scenarios is discussed.

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Recently, much attention is paid to the various properties of the absorption and emission problems in the higher-dimensional black holes. This is mainly due to the fact that the brane-world scenarios [1–3] opens the possibility to make tiny black holes in the future colliders [4–7] by high-energy scattering. For the non-rotating black holes the complete absorption and emission spectra are calculated numerically in Ref. [8], where the effect of the number of extra dimensions  $n$  and the inner horizon radius  $r_-$  on the spectra is carefully examined. It has been found in Ref. [8] that the presence of  $r_-$  generally enhances the absorptivity and suppresses the emission rate while the presence of  $n$  reduces the absorptivity and increases the emission rate regardless of the brane-localized and bulk fields. In fact, this fact can be deduced by considering the Hawking temperature or by computing the effective potential generated by the horizon structure.

Also, the ratio of the low-energy absorption cross section for the Dirac fermion to that for the scalar field is derived analytically [9] in the charged black hole background. For the case of the bulk fields this ratio factor becomes

$$\gamma^{BL} \equiv \frac{\sigma_F^{BL}}{\sigma_S^{BL}} = 2^{-(n+3)/(n+1)} \left[ 1 - \left( \frac{r_-}{r_+} \right)^{n+1} \right]^{(n+2)/(n+1)} \quad (1)$$

and for the case of the brane-localized fields this factor becomes

$$\gamma^{BR} \equiv \frac{\sigma_F^{BR}}{\sigma_S^{BR}} = 2^{(n-3)/(n+1)} \left[ 1 - \left( \frac{r_-}{r_+} \right)^{n+1} \right]^{2/(n+1)}. \quad (2)$$

In the Schwarzschild limit ( $r_- \sim 0$ )  $\gamma^{BL} \sim 2^{-(n+3)/(n+1)}$  and  $\gamma^{BR} \sim 2^{(n-3)/(n+1)}$  which reduces to  $1/8$  when  $n = 0$ , which was derived by Unruh long ago [10]. It is interesting to note that  $\sigma_F^{BL} = \sigma_S^{BL}/2$  and  $\sigma_F^{BR} = 2\sigma_S^{BR}$  when  $n = \infty$ . In the extremal limit ( $r_- \sim r_+$ ) the low-energy absorption cross sections for the brane-localized and bulk fermions goes to zero. The explicit  $n$ -dependence of these ratio factors may play important role in the experimental proof on the existence of the extra dimensions.

Recently, there is a controversy in the question of whether the higher-dimensional black holes radiate mainly on the brane or in the bulk. Ref. [11,12] argued that the Hawking radiation into the bulk is dominant compared to the brane-emission. The main reason for

this is because of the fact that for the tiny black holes the Hawking temperature is much larger than the mass of the light Kaluza-Klein modes. However, Ref. [13] argued that the Hawking radiation on the brane is dominant because the radiation into the bulk by the light Kaluza-Klein modes is strongly suppressed by the geometrical factor. This argument is supported numerically by Ref. [8] when  $n$  is not too large in the charged black holes.

However, the situation can be completely different when the black hole background has an angular momentum. For the rotating black holes the incident waves can be scattered backward with extraction of the black hole's rotating energy, which is called superradiance. The effect of the superradiance in the 4-dimensional Kerr black hole was discussed in Ref. [14–17]. In this context Ref. [18,19] argued that the conventional claim that *the black holes radiate mainly on the brane* can be changed if the effect of the superradiance is involved. In fact, the existence the superradiance was proved analytically [20] and numerically [21,22]. Also the general condition for the existence of the superradiance modes for the scalar, electromagnetic and gravitational waves was derived in Ref. [23] using the Bekenstein argument [24] when the black hole has a single angular momentum parameter.

Here, we would like to extend Ref. [23] to the rotating black holes which have multiple angular momentum parameters. This is important because the tiny rotating black holes that will be produced in the future colliders due to the nonzero impact parameter can have multiple components of the angular momentum since the brane thickness is of order of  $1/\text{TeV}$ .

We start with a spacetime of the  $(N + 1)$ -dimensional rotating black hole derived by Myers and Perry in Ref. [25]:

$$ds^2 = -dt^2 + \sum_{i=1}^{N/2} (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r^2}{\Pi \mathcal{F}} \left( dt + \sum_{i=1}^{N/2} a_i \mu_i^2 d\phi_i \right)^2 + \frac{\Pi \mathcal{F}}{\Pi - \mu r^2} dr^2 \quad (3)$$

where

$$\mathcal{F} = 1 - \sum_{i=1}^{N/2} \frac{a_i^2 \mu_i^2}{(r^2 + a_i^2)} \quad (4)$$

$$\Pi = \prod_{i=1}^{N/2} (r^2 + a_i^2).$$

In Eq.(3) we assumed that  $N$  is even. The odd  $N$  case will be discussed later. The  $\mu_i$  are not all independent but obeys

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_{N/2}^2 = 1. \quad (5)$$

The mass  $M$  and angular momenta  $J_i$  of the black hole (3) are

$$M = \frac{(N-1)\Omega_{N-1}}{16\pi G}\mu \quad J_i = \frac{2}{N-1}Ma_i \quad (i = 1, 2, \dots, \frac{N}{2}) \quad (6)$$

where  $\Omega_{N-1} = 2\pi^{N/2}/\Gamma[N/2]$  is the area of a unit  $(N-1)$ -sphere and  $G$  is a  $(N+1)$ -dimensional Newton constant, which will be assumed to be unity from now on.

Now, we would like to calculate the horizon area  $A$  of the spacetime (3), which is given by

$$A = \int_0^{2\pi} d\phi_1 \cdots d\phi_{\frac{N}{2}} \int_0^1 d\mu_1 \int_0^{\sqrt{1-\mu_1^2}} d\mu_2 \cdots \int_0^{\sqrt{1-\mu_1^2-\cdots-\mu_{\frac{N}{2}-2}^2}} d\mu_{\frac{N}{2}-1} \sqrt{\det M} \quad (7)$$

where  $\det M = \det(g_{\mu_i, \mu_j}) \det(g_{\phi_i, \phi_j})|_{r=r_H}$ . The horizon radius  $r_H$  is defined by solving

$$\prod_{i=1}^{N/2} (r_H^2 + a_i^2) = \mu r_H^2. \quad (8)$$

The factorization of  $\det M$  comes from  $g_{\mu_i, \phi_j} = 0$ . It is not difficult to show  $\det M = \mu^2 r_H^2 \mu_1^2 \mu_2^2 \cdots \mu_{N/2-1}^2$ , which makes  $A$  in the following simple form

$$A = \Omega_{N-1} \mu r_H. \quad (9)$$

Now, we regard  $M$  and  $J_i (i = 1, 2, \dots, N/2)$  are independent variables. Then one can write

$$dA = \frac{\partial A}{\partial M} dM + \sum_{i=1}^{N/2} \frac{\partial A}{\partial J_i} dJ_i. \quad (10)$$

Firstly, let us compute  $\partial A / \partial M$ , which is

$$\frac{\partial A}{\partial M} = \Omega_{N-1} \left[ \frac{\mu}{M} r_H + \mu \frac{\partial r_H}{\partial M} \right]. \quad (11)$$

To compute  $\partial r_H / \partial M$  we use Eq.(8). Differentiating Eq.(8) with respect to  $M$ , one can show easily

$$\frac{\partial r_H}{\partial M} = \frac{1}{\mathcal{B}} \left[ \frac{r_H}{2M} + \frac{r_H}{M} \sum_{i=1}^{N/2} \frac{a_i^2}{r_H^2 + a_i^2} \right] \quad (12)$$

where

$$\mathcal{B} = \sum_{i=1}^{N/2} \frac{r_H^2}{r_H^2 + a_i^2} - 1. \quad (13)$$

It is important to note that  $\mathcal{B}$  can be expressed as

$$\mathcal{B} = r_H \kappa \quad (14)$$

where  $\kappa$  is a surface gravity defined

$$\kappa = \left. \frac{\partial_r \Pi - 2\mu r}{2\mu r^2} \right|_{r=r_H}. \quad (15)$$

Since the surface gravity is proportional to the Hawking temperature, we can assume  $\mathcal{B} > 0$ .

Inserting Eq.(12) into (11) and using Eq.(6) yields

$$\frac{\partial A}{\partial M} = \frac{8\pi r_H}{\mathcal{B}}. \quad (16)$$

Next, let us compute  $\partial A / \partial J_j$ , which is

$$\frac{\partial A}{\partial J_j} = \Omega_{N-1} \mu \frac{\partial r_H}{\partial J_j}. \quad (17)$$

Differentiating Eq.(8) with respect to  $J_j$ , one can show easily

$$\frac{\partial r_H}{\partial J_j} = -\frac{(N-1)r_H}{2M\mathcal{B}} \Omega_j \quad (18)$$

where

$$\Omega_j = \frac{a_j}{r_H^2 + a_j^2} \quad (19)$$

is a frequency of the black hole arising due to the  $j$ -th angular momentum  $J_j$ . Inserting Eq.(18) into (17) yields

$$\frac{\partial A}{\partial J_j} = -\frac{8\pi r_H}{\mathcal{B}} \Omega_j. \quad (20)$$

Thus inserting (16) and (20) into (10) simply yields

$$dA = \frac{8\pi r_H}{\mathcal{B}} \left[ dM - \sum_{i=1}^{N/2} \Omega_i dJ_i \right]. \quad (21)$$

Bekenstein has shown in Ref. [24] that for scalar, electromagnetic, and gravitational waves  $dJ_i/dM$  is expressed in terms of the stress-energy tensor  $T_{\mu\nu}$  as following

$$\frac{dJ_i}{dM} = -\frac{T_{\phi_i}^r}{T_t^r} \quad (22)$$

where  $\phi_i$  are the azimuthal angles associated with  $J_i$ . Since the incident waves should have the factorization factors  $e^{im_i\phi_i}e^{-i\omega t}$ , one can show easily that Eq.(22) reduces to

$$\frac{dJ_i}{dM} = \frac{m_i}{\omega} \quad (23)$$

where  $m_i$  and  $\omega$  are the azimuthal quantum numbers corresponding to  $\phi_i$  and energy of the incident waves respectively. Inserting (23) into (21), one can show easily

$$dA = \frac{8\pi r_H}{\mathcal{B}} dM \left[ 1 - \frac{1}{\omega} \sum_{i=1}^{N/2} m_i \Omega_i \right]. \quad (24)$$

Since  $A/4$  is a black hole entropy, Eq.(24) gives a condition

$$dM \left[ 1 - \frac{1}{\omega} \sum_{i=1}^{N/2} m_i \Omega_i \right] > 0. \quad (25)$$

Since the existence of the superradiance modes implies  $dM < 0$ , it is easy to show that the condition for the existence of the superradiance is

$$0 < \omega < \sum_{i=1}^{N/2} m_i \Omega_i. \quad (26)$$

In Ref. [20] the condition for the superradiance for the incident scalar wave was shown to be  $0 < \omega < m\Omega_a + k\Omega_b$ , which is manifestly special case of Eq.(26) in  $N = 4$ . Furthermore, our conclusion (26) holds not only for the scalar wave but also for the electromagnetic and gravitational waves.

For a completeness we consider the odd  $N$  case. In this case the metric for the rotating black hole is changed into

$$\begin{aligned}
ds^2 = & -dt^2 + r^2 d\alpha^2 + \sum_{i=1}^{(N-1)/2} (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) \\
& + \frac{\mu r}{\Pi \mathcal{F}} \left[ dt + \sum_{i=1}^{(N-1)/2} a_i \mu_i^2 d\phi_i \right]^2 + \frac{\Pi \mathcal{F}}{\Pi - \mu r} dr^2
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\Pi &= \prod_{i=1}^{(N-1)/2} (r^2 + a_i^2) \\
\mathcal{F} &= 1 - \sum_{i=1}^{(N-1)/2} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}
\end{aligned} \tag{28}$$

and

$$\sum_{i=1}^{(N-1)/2} \mu_i^2 + \alpha^2 = 1. \tag{29}$$

The mass  $M$  and the angular momenta  $J_i$  of the black hole (27) are same with Eq.(6). The horizon area is slightly different from Eq.(9):

$$\begin{aligned}
A &= \int_0^{2\pi} d\phi_1 \cdots d\phi_{\frac{N-1}{2}} \int_0^1 d\mu_1 \int_0^{\sqrt{1-\mu_1^2}} d\mu_2 \cdots \int_0^{\sqrt{1-\mu_1^2-\cdots-\mu_{\frac{N-3}{2}}^2}} d\mu_{\frac{N-1}{2}} \\
&\quad \times \sqrt{\det(g_{\mu_i \mu_j}) \Big|_{r=r_H} \det(g_{\phi_i \phi_j}) \Big|_{r=r_H}} \\
&= \frac{\Omega_{N-1}}{2} \mu r_H.
\end{aligned} \tag{30}$$

Then it is straightforward to show

$$\frac{\partial A}{\partial M} = \frac{8\pi r_H}{\mathcal{B}} \quad \frac{\partial A}{\partial J_i} = -\frac{8\pi r_H}{\mathcal{B}} \Omega_i \tag{31}$$

where  $\Omega_i = a_i/(r_H^2 + a_i^2)$  are the rotational frequency of the black hole and

$$\mathcal{B} = \frac{2r_H}{\mu} \sum_{i=1}^{(N-1)/2} \prod_{j \neq i} (r_H^2 + a_j^2) - 1. \tag{32}$$

One can show easily  $\mathcal{B} = 2r_H \kappa$  where  $\kappa$  is a surface gravity defined

$$\kappa = \left. \frac{\partial_r \Pi - \mu}{2\mu r} \right|_{r=r_H}. \tag{33}$$

Then it is easy to show

$$dA = \frac{8\pi r_H}{\mathcal{B}} \left[ dM - \sum_{i=1}^{(N-1)/2} \Omega_i dJ_i \right]. \quad (34)$$

Following the same procedure we can conclude that the condition for the existence of the superradiance modes for the incident scalar, electromagnetic and gravitational waves reduces to

$$0 < \omega < \sum_{i=1}^{(N-1)/2} m_i \Omega_i \quad (35)$$

where  $m_i$  are the azimuthal quantum numbers.

In this paper we derived the condition for the existence of the superradiance modes for the incident scalar, electromagnetic and gravitational waves when the spacetime background is a  $(N+1)$ -dimensional rotating black holes with multiple angular momentum parameters. Our final condition reduces to a simple form  $0 < \omega < \sum_i m_i \Omega_i$ , where  $m_i$  is an azimuthal quantum numbers of the incident waves associated with the  $i$ -th angle  $\phi_i$  and  $\Omega_i$  is a rotating frequency corresponding to the  $i$ -th angular momentum  $J_i$ . In  $4d$  Kerr black hole it is well-known that there is no superradiance mode for the incident fermionic wave [26,27]. It is of interest to check whether this property is maintained in the higher-dimensional rotating black hole background or not. Another interesting point arising due to the existence of the superradiance modes in the rotating brane-world black holes is that the standard claim ‘*the black holes radiate mainly on the brane*’ is not obvious in this background. Thus, it is necessary to check which one is dominant between the bulk-emission and the brane-emission by adopting an appropriate numerical method. We hope to report this issue elsewhere.

**Acknowledgement:** This work was supported by the Korea Research Foundation under Grant (KRF-2003-015-C00109).

## REFERENCES

- [1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *The Hierarchy Problem and New Dimensions at a Millimeter*, Phys. Lett. **B429** (1998) 263 [hep-ph/9803315].
- [2] L. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *New Dimensions at a Millimeter to a Fermi and Superstrings at a TeV*, Phys. Lett. **B436** (1998) 257 [hep-ph/9804398].
- [3] L. Randall and R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221].
- [4] S. B. Giddings and T. Thomas, *High energy colliders as black hole factories: The end of short distance physics*, Phys. Rev. **D65** (2002) 056010 [hep-ph/0106219].
- [5] S. Dimopoulos and G. Landsberg, *Black Holes at the Large Hadron Collider*, Phys. Rev. Lett. **87** (2001) 161602 [hep-ph/0106295].
- [6] D. M. Eardley and S. B. Giddings, *Classical black hole production in high-energy collisions*, Phys. Rev. **D66** (2002) 044011 [gr-qc/0201034].
- [7] D. Stojkovic, *Distinguishing between the small ADD and RS black holes in accelerators*, Phys. Rev. Lett. **94** (2005) 011603 [hep-ph/0409124].
- [8] E. Jung and D. K. Park, *Absorption and Emission Spectra of an higher-dimensional Reissner-Nordström black hole* [hep-th/0502002].
- [9] E. Jung, S. H. Kim and D. K. Park, *Ratio of absorption cross section for Dirac fermion to that for scalar in the higher-dimensional black hole background* [hep-th/0503027].
- [10] W. G. Unruh, *Absorption cross section of small black holes*, Phys. Rev. **D14** (1976) 3251.
- [11] P. Argyres, S. Dimopoulos and J. March-Russell, *Black Holes and Sub-millimeter Dimensions*, Phys. Lett. **B441** (1998) 96 [hep-th/9808138].

- [12] T. Banks and W. Fischler, *A Model for High Energy Scattering in Quantum Gravity* [hep-th/9906038].
- [13] R. Emparan, G. T. Horowitz and R. C. Myers, *Black Holes radiate mainly on the Brane*, Phys. Rev. Lett. **85** (2000) 499 [hep-th/0003118].
- [14] Y. B. Zel'dovich, *Generation of waves by a rotating body*, JETP Lett. **14** (1971) 180.
- [15] W. H. Press and S. A. Teukolsky, *Floating Orbits, Superradiant Scattering and the Black-hole Bomb*, Nature **238** (1972) 211.
- [16] A. A. Starobinskii, *Amplification of waves during reflection from a rotating black hole*, Sov. Phys. JETP **37** (1973) 28.
- [17] A. A. Starobinskii and S. M. Churilov, *Amplification of electromagnetic and gravitational waves scattered by a rotating black hole*, Sov. Phys. JETP **38** (1974) 1.
- [18] V. Frolov and D. Stojković, *Black hole radiation in the brane world and the recoil effect*, Phys. Rev. **D66** (2002) 084002 [hep-th/0206046].
- [19] V. Frolov and D. Stojković, *Black Hole as a Point Radiator and Recoil Effect on the Brane World*, Phys. Rev. Lett. **89** (2002) 151302 [hep-th/0208102].
- [20] V. Frolov and D. Stojković, *Quantum radiation from a 5-dimensional black hole*, Phys. Rev. **D67** (2003) 084004 [gr-qc/0211055].
- [21] D. Ida, K. Oda and S. C. Park, *Anisotropic scalar field emission from TeV scale black hole* [hep-ph/0501210].
- [22] C. M. Harris and P. Kanti, *Hawking Radiation from a  $(4 + n)$ -Dimensional Rotating Black Hole* [hep-th/0503010].
- [23] E. Jung, S. H. Kim and D. K. Park, *Condition for Superradiance in Higher-dimensional Rotating Black Holes* [hep-th/0503163].

- [24] J. D. Bekenstein, *Extraction of Energy and Charge from a Black Hole*, Phys. Rev. **D7** (1973) 949.
- [25] R. C. Myers and M. J. Perry, *Black Holes in Higher Dimensional Space-Times*, Ann. Phys. **172** (1986) 304.
- [26] W. Unruh, *Separability of the Neutrino Equations in a Kerr Background*, Phys. Rev. Lett. **31** (1973) 1265.
- [27] S. Chandrasekhar, *The Mathematical Theory of Black Hole* (Oxford University Press, New York, 1983).